International Journal of Theoretical Physics, Vol. 4, No. 4 (1971), pp. 305–320

On the Geometric Structure of a Parity-Conserving Relativistic Quantum Field Theory

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Received: 18 *January* 1971

Abstract

Wheeler's conjecture that there might exist a 'principle' which rules out parity-non-conserving spaces is analysed. The following result has been obtained: A local relativistic quantum field theory is parity-conserving if the following conditions hold:

(a) The fields are derived from geometry, i.e. they are represented by quantised currents (in the sense of de Rham); and
(b) The theory may be defined on a connected and, under certain restrictions, on a disconnected orientable space-time continuum M^4.

1. *Introduction*

Invariance of the physical laws under space reflections is equivalent to the indiscernability of right and left. Such an equivalence would entail the non-existence of any internal physical or geometrical structure which would permit an absolute distinction between right- and left-handed coordinate systems to be made. However, there exist experiments, such as the circular polarisation of electrons emitted in β-decay, which give strong evidence for a certain handedness. This kind of asymmetrical phenomenon does not imply, however, the failure of the geometrical equivalence of right and left. One would rather have to conjecture a certain 'screw-sense' in the dynamical laws. The question whether it is the geometry or the dynamical law that must have a 'twisted' structure may be analyzed by means of the following example: A magnetic needle placed below a current-carrying wire will be deflected in a certain sense, according to Ampère's corkscrew-rule. But this rule, also referred to as right-hand rule, makes strictly no sense if one does not admit a universal 'right' and 'left'. Mathematically this means that the underlying three-manifold must be oriented (refer to Section 3). Otherwise stated: It would be the geometry which

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would not be 'twisted', i.e. endowed with any screw-like structure. (Twisted structures are, for instance, provided by the so-called Möbius strip or the Klein bottle.)

On the other hand, the electromagnetic field is characterised, within a relativistic framework, by the two-form (refer to Section 3)

$$\omega = \sum_{\mu,\nu} F_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ & 0 & -H_3 & H_2 \\ & & 0 & -H_1 \\ & & & 0 \end{pmatrix}; \qquad F_{\mu\nu} = -F_{\nu\mu}$$

Since this form represents a polar *and* an axial vector, the underlying four-dimensional space-time continuum need not necessarily be orientable. However, the above-mentioned experiment of β-decay indicates that one has to expect the universe to be orientable. Indeed, the probability of electron-emission is greater for one of the two half-spaces, defined by a plane, which is perpendicular with respect to the axis of rotation. Such a phenomenon is obviously independent of any space orientation. Thus one may define some privileged orientation of this space. This space would have to be orientable.

Wheeler (1962) has suggested that there might exist a 'principle' which rules out non-orientable or parity-non-conserving spaces. This problem will be analysed in the subsequent sections of this paper.

2. *Discussion of the Conventional Parity-Conserving Conditions*

Our goal is to approach a geometrical quantum field theory in the sense of Wheeler & Misner (1957), where the elementary particles, i.e. the quantum fields, are considered to be derived from geometry and not to be added to geometry. We exhibit in this section the set-up of the parity-conserving conditions within the framework of a conventional quantum field theory. These will be transcribed in our subsequent Section 5, into an appropriate geometrical language.

Let $\Phi_1(x),\ldots, \Phi_n(x)$ and $\Psi_1(x),\ldots, \Psi_m(x)$ be selfadjoint Bose-Einstein and Fermi-Dirac fields respectively which are supposed to satisfy the following canonical equal-time commutation and anti-commutation rules (together with the correct connection of spin and statistics):

$$\begin{aligned} &[\Psi(x), \Psi(x')]_{+x^0=x^{0\prime}} = [\tilde{\Psi}(x), \tilde{\Psi}(x')]_{+x^0=x^{0\prime}} = 0 \\ &[\Psi(x), \tilde{\Psi}(x')]_{+x^0=x^{0\prime}} = \gamma^0\, \delta(\mathbf{x}-\mathbf{x}') \\ &\left[\Phi(x), \frac{\partial \Phi}{\partial x^0}(x')\right]_{-x^0=x^{0\prime}} = i\hbar c\, \delta(\mathbf{x}-\mathbf{x}') \qquad x = (x^0, \mathbf{x}) \quad (2.1) \\ &[\Psi(x), \Phi(x')]_{-x^0=x^{0\prime}} = [\tilde{\Psi}(x), \Phi(x')]_{-x^0=x^{0\prime}} = 0 \end{aligned}$$

A quantum field theory which is described by these rules may be characterised to be space-inversion invariant by means of the following:

Definition 1: A theory of type (2.1) is said to be invariant under a space-inversion $P \in L_-^\uparrow$ if there exists a unitary operator $U(P)$ such that conditions (2.2)–(2.4) hold:

$$U(P)\,\Phi(\phi)\,U(P)^{-1} = \Phi[(0,P)\,\phi] \tag{2.2}$$

$$U(P)\,\Psi(\phi)\,U(P)^{-1} = +\gamma^0\,\Psi[(0,P)\,\phi] \tag{2.3a}$$

where

$$P : (x^0, \mathbf{x}) \to (x^0, -\mathbf{x})$$

$$(0,P)\,\phi(x) = \phi(P^{-1}\,x) = \phi(x^0, -\mathbf{x})$$

and

$$\Phi(\phi) = \int dx \phi(x)\,\Phi(x) \tag{2.3b}$$

$$\Psi(\phi) = \int dx \phi(x)\,\Psi(x) \tag{2.3c}$$

are the fields $\Phi(x)$ and $\Psi(x)$ which are averaged with testing functions

$$[U(P), H] = 0 \qquad \phi \in \mathscr{D} = \{\phi | \operatorname{supp} \phi : \text{compact } \phi \in C^\infty\} \tag{2.4}$$

Remark 1: The transformation law

$$\Psi(x) \to -\gamma^0\,\Psi(Px) \tag{2.3'a}$$

for spin fields is just as good a definition of the parity operation as (2.3a).

Indeed: if $U(P)$ is the transformation which produces the first choice, then $U(R)\,U(P)$ will produce the second, if $R \in SO(3)$ rotates through an angle 2π. A permissible choice for γ^0 is

$$\gamma^0 = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & 0 & -1 \\ & & 1 & 0 \end{array}\right)$$

Remark 2: Since fields in a single point are not observable as stressed by Bohr & Rosenfeld (1933), only averaged field operators are well-defined, i.e. more precisely than their corresponding Wightman distributions.

Remark 3: Relation (2.4) is to be understood in the sense that, even though one upholds the intrinsic geometrical equivalence of right and left, one cannot reduce all the concepts, which are related to a dynamical law, to geometrical notions. Our subsequent discussion of the equation $[H, U(P)] = 0$ in Section 6 provides the *dynamical* feature of the fact that the theory cannot distinguish between left and right. This dynamical feature originates from the fact that the selfadjoint operator H, commonly referred to as the dynamical operator, provides the information on the time-development of a system, through

$$\Phi(x) = \exp(iHx^0)\,\Phi(\mathbf{x})\exp(-iHx^0) \tag{2.5}$$

Remark 4: Since we shall be dealing in Section 4 with fields which are derived from geometry, statements about coupling constants within such a framework are devoid of any meaning, as stressed by Wheeler (1962). Therefore one gets into difficulties by considering Hamiltonians, such as

$$H = H(g_1, \ldots, g_n) \qquad g_k : \text{coupling constants} \tag{2.6}$$

We shall circumvent these difficulties by adopting the following point of view.

Instead of making the interaction responsible for P-violation, one may assume that space-inversion invariance does not hold already for free neutrinos (the above-mentioned fields $\Psi(x)$ are then to be considered as Weyl fields). This screw character of the Weyl neutrino is readily shown to imply parity violation in β-decay and other processes quite irrespective of the form of interaction. Therefore our subsequent reasonings will hold for any type of interacting quantum field theory. Instead of discussing P-violation within the framework of interacting fields, one may, by virtue of the aforementioned facts, base this discussion on the asymptotic free fields

$$\Psi_{\substack{\text{in}\\ \text{out}}}(x) = \lim_{t\to\mp\infty} \Psi(x) \tag{2.7}$$

We shall make use of this possibility.

3. *The Basic Set-Up*

Let $M = M^4$ be a differentiable manifold of dimension 4 and class C^∞, which represents the space-time continuum. M^4 is supposed to be endowed with a pseudo-Riemannian structure whose metric $ds^2 = g_{\mu\nu}(x^\mu)\,dx^\mu\,dx^\nu$ ($(x^\mu : \mu = 0, 1, 2, 3)$ denote the admissible local coordinates) is of the hyperbolic normal type. Special relativity is taken into account by the introduction of the principal fibre bundle $E(M^4)$ over the base space M^4. This fibre bundle is defined as follows:

$$E(M^4) = \{(x, \rho_x) : x \in M^4\} \tag{3.1}$$

With respect to a set of orthonormalised frames $^{\rho_x}$ which characterise also $E(M^4)$, the metric can be written on an open neighbourhood of M^4:

$$ds^2 = g_{\mu\nu}\,\theta^\mu\,\theta^\nu = (\theta^0)^2 - \sum_k (\theta^k)^2 \tag{3.2}$$

where the θ^k denote Pfaffians. The structural group of $E(M^4)$ is the complete homogeneous Lorentz group $\{(0, \Lambda)\}$.

An orientation, i.e. a total orientation of M^4 (this total orientation appears to be the product of a spatial and time orientation) is a pseudo-scalar ϵ of square 1. If the manifold M^4 is orientable, such a geometrical object exists and is defined by one component: $\epsilon = \mp 1$.

The following definition makes this more precise:

Definition 2: A continuous system of orientations of M^4 is a continuous function

$$\epsilon : x \to \{-1, +1\} \qquad \forall x \in M^4 \tag{3.3}$$

That is: $\epsilon(x)$ is continuous in $x \in M$ implies this function to be constant in some neighbourhood $U(x)$ of x. Definition (2.2) can be made more explicit as follows: Let (U, Φ) be a local chart of M^4 and $D\Phi(x) \equiv \Phi'(x)$ the Fréchet derivative in $x \in U$. The Isomorphism

$$D\Phi(x) : T_x(M) \to \mathbf{R}^n \qquad \text{(in our case } \mathbf{R}^4\text{)} \tag{3.4}$$

i.e.

$$(e_1, \ldots, e_n) \to (D\Phi(x)^{-1} e_1, \ldots, D\Phi(x)^{-1} e_n)$$

which are ordered bases in $\mathbf{R}^n$ and T_x respectively provides a canonical correspondence between the classes of bases of $\mathbf{R}^4$ and $T_x(M)$. One has:

$$\epsilon(x, \Phi) := \epsilon(x) = +1$$

if Φ establishes a correspondence between a positive (negative) class of bases of $\mathbf{R}^4$ with a positive (negative) class of bases of $T_x(M)$.

$$\epsilon(x) = -1$$

if Φ establishes a correspondence between classes of bases of opposite signs. Thus one can give the following:

Definition 3: The manifold M^4 is said to be orientable if it possesses at least one continuous system of orientations. Furthermore, M^4 is oriented if one chooses one system of orientations.

This definition can easily be shown to be equivalent to the classical definition of orientability, which states:

Definition 3′: A differentiable manifold M^n is said to be orientable if there exists an atlas $(U_i, \Phi_i)_{i \in I}$ such that, for each intersection $U_i \cap U_j$ of local charts, one has

$$dx^1 \wedge \ldots \wedge dx^n = J\, dx^{1'} \wedge \ldots \wedge dx^{n'} \tag{3.6}$$

where $J = \det(\partial x^i / \partial x^{j'}) > 0$.

In order to re-express a quantum field theory in a purely geometrical manner, we make use of a calculus of quantised differential forms (Section 4), which has been extensively studied by Segal (1968) and Lichnerowicz (1964), thus extending the conventional theory of differential forms as outlined below. The motivation of such a programme is that the calculus of exterior differential forms, already in its conventional framework, provides a much deeper insight into geometrical and physical laws than tensor methods. This has been displayed in a fundamental paper by Wheeler & Misner (1957), in which exterior forms provide an adequate framework for classical physics. Further applications of exterior forms in classical physics are exhibited in the book of Mme. Choquet-Bruhat (1968) as well as by Flanders (1963) and von Westenholz (1970).

We introduce two classes of exterior forms: the normal or even ones and the twisted or odd differential forms. The general set-up for these forms is the following:

Let M be a differentiable manifold of class $k \geqslant 1$ and dimension n. A differentiable form of degree p on M is a differentiable cross-section ω of class C^l $(l \leqslant k-1)$ of the exterior fibre bundle

$$\Lambda^p(M) = \bigcup_{x \in M} \Lambda^p(T_x{}^*) \tag{3.7}$$

over M, which represents a differentiable manifold of class $k-1$.

Here, M denotes the base space and $\Lambda^p(T_x{}^*)$ the fibres endowed with the natural differentiable structure C^{k-1}, in particular, $\Lambda^1(T_x{}^*) := T_x{}^*$ denotes the dual space of the tangent space T_x at $x \in M$ and $\Lambda^0(T_x{}^*) = \mathbf{R}$. Thus a differential form ω of degree p and class C^l may also be characterised as an element

$$\omega : x \to \omega(x) \in \Lambda^p(T_x{}^*) \tag{3.8}$$

whose local representation is given by

$$\omega = \sum_{i_1 < \ldots < i_p} a_{i_1, \ldots, i_p} dx^{i_1} \ldots dx^{i_p} \tag{3.9}$$

We define addition and exterior multiplication of exterior polynomials according to the formulas

$$(\omega + \omega')(x) = \omega(x) + \omega'(x) \tag{3.10a}$$

$$(\omega \wedge \omega')(x) = \omega(x) \wedge \omega'(x) \tag{3.10b}$$

(3.10b) is a bilinear, associative and anti-commutative operation, i.e., if $\omega \in F^p$ and $\omega' \in F^q$, then

$$\omega \wedge \omega' = (-1)^{pq} \omega' \wedge \omega \tag{3.11}$$

is valid. Endowed with this law, one defines the following graded module:

$$F = F^0 \oplus F^1 \oplus \ldots \oplus F^n \tag{3.12}$$

Our next task is to associate differential forms with non-orientable manifolds. According to de Rham (1960), the most natural tools related to this class of manifolds are the so-called twisted differential forms. These may be characterised as follows:

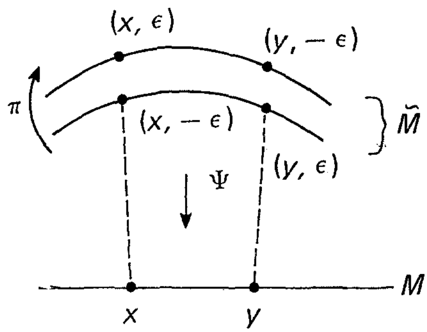
Let $\tilde{M}$ be the canonically oriented two-sheeted covering space of M:

$$\tilde{M} = \{(x, \epsilon) | x \in M;\ \epsilon \text{ one of the two possible orientations of } T_x \text{ in } x \in M\} \tag{3.13}$$

Thus, to each point $x \in M$, there correspond two points of $\tilde{M}$: $(x, +\epsilon)$ and $(x, -\epsilon)$ (see figure). Otherwise stated: $\tilde{M}$ is a fibre bundle, consisting of the base-space M, the covering projection

$$\Psi : \tilde{M} \to M$$

and a fibre over each point $x \in M$, which is canonically isomorphic to the two-element set consisting of the orientations of T_x at this point.



Let π denote the permutation-map of the two sheets:

$$\begin{aligned} \pi: \tilde{M} &\to \tilde{M} \\ (x, \pm\epsilon) &\to (x, \mp\epsilon) \end{aligned} \tag{3.14}$$

This mapping of the differentiable manifold $\tilde{M}$ into itself induces a mapping π^* of the modules of exterior p-forms on $\tilde{M}$:

$$\begin{aligned} \pi^*: F^p(\tilde{M}) &\to F^p(\tilde{M}) \\ \tilde{\omega} &\to \tilde{\omega}' \end{aligned} \tag{3.15}$$

Therefore, a twisted exterior form $\boldsymbol{\omega}$ on M is defined by the following property of the corresponding form $\tilde{\omega}$ on $\tilde{M}$:

$$\pi^* \tilde{\omega} = -\tilde{\omega} \tag{3.16}$$

whereas a normal exterior form $\omega \in F^p(M)$ is referred to as being invariant under the automorphism π^*, i.e.

$$\begin{gathered} \pi^* \tilde{\omega} = \tilde{\omega}; \qquad \tilde{\omega} \in F^p(\tilde{M}), \qquad \tilde{\omega} = \Psi^* \omega \\ \Psi^*: F^p(M) \to \{\tilde{\omega} \in F(\tilde{M}) : \pi^* \tilde{\omega} = \tilde{\omega}\} \end{gathered} \tag{3.17}$$

Since twisted forms are defined on orientable as well as non-orientable manifolds (de Rham, 1960), the integral

$$\int_M \boldsymbol{\omega} = \tfrac{1}{2} \int_{\tilde{M}} \tilde{\omega} \tag{3.18}$$

always makes sense. On the contrary, if M is non-orientable, one has, by definition,

$$\int_M \omega = 0 \tag{3.19}$$

since normal exterior forms are only defined on orientable manifolds.

Finally, we introduce the notion of a current in the sense of de Rham (1960): Consider again some n-manifold. A linear and continuous functional $\mathbf{T}(\phi)$ on the vector space of all even $(n-p)$-forms (Schwartz, 1957)

$$\mathscr{D}^{n-p} = \{\phi \in C^\infty | \operatorname{supp} \phi : \text{compact}\} \tag{3.20}$$

is called a twisted p-current $\in \mathscr{D}^{p'}$. Let $\mathscr{D}^{n-p}$ be the vector space of twisted $(n-p)$-forms. A linear continuous functional $T \in \mathscr{D}^{p'}$ on $\mathscr{D}^{n-p}$ is called a normal current. A classification of normal and twisted currents is of interest with respect to quantized differential forms (Section 4) and will be given there.

4. *Local Quantised Differential Forms*

The following characterisation of local quantised differential forms is similar to that given by Segal (1968). It extends the notion of conventional differential form in a fashion, bringing it in close relation to the algebra of operators on the manifold M^4.

Definition 4: A quantised differential form over M^4 of degree $p \leqslant 4$ is an alternating p-mapping $M^4 \xrightarrow{\text{into}} \mathscr{L}(\mathscr{H}, \mathscr{H})$, the algebra of linear operators on the Hilbert space $\mathscr{H}$.

Thus the local representation (3.9) may be generalised as follows:

$$\omega = \sum_{i_1 < \ldots < i_p} A_{i_1, \ldots, i_p}(x)\, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \tag{4.1}$$

where

$$\begin{aligned} A_{i_1, \ldots, i_p}(x) &= A(x, e_{i_1}, \ldots, e_{i_p}) \\ &=: A \end{aligned} \tag{4.2}$$

represents a tensor operator $\in \mathscr{L}(\mathscr{H}, \mathscr{H})$ $\forall p$-tuple $(i_1, \ldots, i_p)$.

In a framework of local quantised differential forms, formula (3.11) must be replaced by relation (4.3) according to

Lemma 1

Let ω and ω' be two quantised differential forms. These forms satisfy relationship (3.11) if and only if the tensor operator coefficients (4.2) commute, i.e.

$$\omega \wedge \omega' = (-)^{pq} \omega' \wedge \omega \Leftrightarrow [A_{i_1,\ldots,i_p}, B_{j_1,\ldots,j_p}] = 0 \qquad \forall (i_1,\ldots,i_p) \quad \text{and} \quad \forall (j_1,\ldots,j_q) \tag{4.3}$$

(in shorthand notation: $[A, B] = 0$).

Proof: Let

$$\omega = \sum_{i_1 < \ldots < i_p} A_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$
$$\omega' = \sum_{j_1 < \ldots < j_q} B_{j_1,\ldots,j_q} dx^{i_1} \wedge \ldots \wedge dx^{j_q}$$

Therefore

$$\omega \wedge \omega' = \sum_{i_1 < \ldots < i_p} \sum_{j_1 < \ldots < j_q} A_{i_1,\ldots,i} B_{j_1,\ldots,j_q} dx^{i_1} \ldots dx^{i_p} dx^{j_1} \ldots dx^{j_q}$$
$$\omega' \wedge \omega = \sum_{i_1 < \ldots < i_p} \sum_{j_1 < \ldots < j_q} B_{j_1,\ldots,j_q} A_{i_1,\ldots,i_p} dx^{i_1} \ldots dx^{i_q} dx^{j_1} \ldots dx^{j_p}$$
$$= (-1)^{pq} \sum \sum B_{j_1,\ldots,j_q} A_{i_1,\ldots,i_p} dx^{i_1} \ldots dx^{i_p} dx^{j_1} \ldots dx^{j_q}$$
$$= (-1)^{pq} \omega \wedge \omega'$$
$$\text{iff } A_{i_1,\ldots,i_p} B_{j_1,\ldots,j_q} = B_{j_1,\ldots,j_q} A_{i_1,\ldots,i_p}$$

In order to take into account spin and statistics as well as to re-state the commutations relations (2.1) in terms of quantised exterior forms, we have to classify tensor- and spinor-exterior forms fields. This can be readily done in agreement with Lichnerowicz (1964) and according to formula (4.2):

I: *Scalar fields*: These are quantised 0-forms $\in F^0(M^4) \equiv \mathscr{D}^0$, that is, simply operators $\in \mathscr{L}(\mathscr{H}, \mathscr{H})$.

II: *Tensor fields*: These are given by the quantised forms ω, $\omega' \ldots$ whose coefficients are the tensor operators (4.2).

III: *Spinor fields*: According to Lichnerowicz (1964), the classical spin fields may be obtained as follows:

Let $F^p(M^4)$ be the module of exterior forms over M^4 and $S(M^4)$ be the module of spinors over M^4. Then there exists a module-isomorphism i between these modules, which assigns to each homogeneous p-form $\overset{p}{\omega}$ a spinor as follows:

$$\Psi = \sum_{p=0}^{4} \frac{1}{p!} \gamma^{i_1}, \ldots, \gamma^{i_p} a_{i_1,\ldots,i_p} = \sum_{p=0}^{4} i\overset{p}{\omega} \tag{4.4}$$

i.e.

$$\Psi = i\omega \qquad \omega = \sum \overset{p}{\omega} \tag{4.5}$$

inhomogeneous form, where

$$\overset{p}{\omega} = \sum_{i_1 < \ldots < i_p} a_{i_1 \ldots i_p} dx^{i_1} \ldots dx^{i_p}$$

γ^{i_1} are the anti-Hermitean 4×4 complex Dirac matrices ($i = 0,1,2,3$) which satisfy the commutation rule $\gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij}I$. The spinor forms (4.4) may be quantised in a straightforward manner.

The generalisation of the averaged fields (2.3b) and (2.3c) is obtained by the use of operator-valued currents and

Lemma 2

There exist operator-valued currents on M^4 which are formally defined as

$$\omega(\boldsymbol{\phi}) = \int \omega \wedge \boldsymbol{\phi} \tag{4.6}$$

where $\omega(x) \in \mathscr{L}(\mathscr{H},\mathscr{H})$ and $\boldsymbol{\phi} \in \mathscr{D}^p$. They constitute a generalisation of the averaged quantised fields of the type $T(\phi) = \int_{\mathbf{R}^4} d^4x\phi(x)T(x)$ (which stands for (2.3b–c)), where $\phi \in \mathscr{D}$ and $T(x) \in \mathscr{L}(\mathscr{H},\mathscr{H})$.

Remark 5: The operator-valued currents (4.6) constitute averaged quantised forms. They make sense only in a framework of generalised Wightman Distributions (4.7) $W(\boldsymbol{\phi}) = (\Phi, \omega(\boldsymbol{\phi})\Psi)$, since only for these does one have the property of continuity, i.e.:

$$\boldsymbol{\phi}_n \to 0 \Rightarrow W(\boldsymbol{\phi}_n) \to 0 \qquad \boldsymbol{\phi}_n \in \mathscr{D}^p \tag{4.7}$$

which in turn generalise the de Rham currents (de Rham (1960).

Proof of Lemma 2: We have to show that, if $M^4 = \mathbf{R}^4$, $\omega \in \mathscr{D}^{n\prime}$ (quantised $\omega \in \mathscr{L}(\mathscr{H},\mathscr{H})$) and $\phi \in \mathscr{D}^0$, $\omega(\boldsymbol{\phi})$ becomes identical with $T(\phi)$. Since $\mathbf{R}^4$ is oriented, $\mathscr{D}^k$ and $\mathscr{D}^k$ are isomorphic $\forall k$. Therefore we may choose fields of the form (4.6) which have the properties

$$\omega(\lambda\boldsymbol{\phi}) = \lambda\omega(\boldsymbol{\phi}), \qquad \omega(\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2) = \omega(\boldsymbol{\phi}_1) + \omega(\boldsymbol{\phi}_2) \tag{4.8}$$

and

$$\omega(\boldsymbol{\phi}) = \int dx^{j_1}, \ldots, dx^{j_{n-k}} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \phi_{i_1, \ldots, i_k}(x)\, \omega_{j_1, \ldots, j_{n-k}}(x) \tag{4.9}$$

which is the representation of (4.6) in local coordinates. In particular, if $\phi \in \mathscr{D}^0 \simeq \mathscr{D}^0$ is a testing function with compact support and $n = 4$, this yields: ω is a four-operator-valued current, i.e. $\omega = \omega_{0,\ldots,3}(x)\,dx^0 \ldots dx^3$ where $\omega_{0,\ldots,3}(x) \in \mathscr{L}(\mathscr{H}, \mathscr{H})$, and therefore

$$\omega(\boldsymbol{\phi}) = \omega(\phi) = \int dx\phi(x^0,\ldots,x^3)\,\omega_{0,\ldots,3}(x) \qquad dx = dx^0 \wedge \ldots \wedge dx^3 \tag{4.10}$$

Formula (4.10) represents the conventional quantum fields.

Remark 6: From formula (4.10) and remark 5, we infer that the physically correct operator-valued currents are given by

$$W(\boldsymbol{\phi}) = (\Phi, \omega(\boldsymbol{\phi})\Psi) = \int \boldsymbol{\phi}(x)(\Phi, \omega_{0,\ldots,3}(x)\Psi)\,dx \qquad \boldsymbol{\phi} \in \mathscr{D}^0 \tag{4.11}$$

where

$$\Phi = \Phi_{0,\ldots,3}(x)\,dx^0 \wedge \ldots \wedge dx^3 \in F^4(M^4)$$

and

$$\Psi = \Psi_{0,\ldots,3}(x)\,dx^0 \wedge \ldots \wedge dx^3 \in F^4(M^4)$$

$$\begin{aligned}\Psi' &= \omega_{0,\ldots,3}(x)\Psi = \omega_{0,\ldots,3}(x)\Psi_{0,\ldots,3}(x)\,dx^0 \ldots dx^3 \\ &= \Psi'_{0,\ldots,3}\,dx^0,\ldots,dx^3 \in F^4(M^4)\end{aligned}$$

Formula (4.11) necessitates the introduction of a Hilbert space of four-forms (de Rham, 1960; Schwartz, 1957) endowed with the scalar product

$$(\Phi, \Psi') = \int\int \ldots \int \Phi\Psi'^* \tag{4.12}$$

which has the properties:

$$(\Phi, \Psi') = (\Psi', \Phi)$$
$$(\Phi, \Phi) \geqslant 0$$

and $(\Phi, \Phi) = 0$ if and only if $\Phi \equiv 0$.

$\Psi'^* \in F^0(M^4)$ denotes the so-called adjoint form of Ψ (de Rham, 1960). This Hilbert space admits the usual Fock space concepts such as the vacuum-state, represented by a constant function 1 which is an 0-form, creation-, annihilation and total number of particles-operator, etc., as exhibited by Souriau (1964).

The generalisation of (4.11) to any form $\phi \in \mathscr{D}^p$ is straightforward but will not be needed in our further discussion.

Remark 7: Besides expression (4.6) for $\omega(\boldsymbol{\phi})$, twisted operator-valued currents such as

$$\boldsymbol{\omega}(\phi) = \int \boldsymbol{\omega} \wedge \phi \tag{4.6'}$$

could have been considered. However, this class of operator-valued currents does not provide an admissible generalisation of the conventional 'smeared' quantum fields. Indeed, these currents are defined by 'twisted operators'

$\omega_{j_1,\ldots,j_{n-p}}(x) \in \mathscr{L}(\mathscr{H},\mathscr{H})$ which obviously have no physical meaning. As an illustrative example of this point, the quantised Maxwell field form $\omega = \sum F^{\mu\nu}(x)\,dx^\mu\,dx^\nu$ accounts for this. Indeed, it defines a normal operator-valued current on $\mathbf{R}^4$.

5. *The Geometric Structure of a Parity-Conserving Theory*

We are now able to analyse Wheeler's question as to whether 'parity-conserving spaces' are ruled out by any principle. Sufficient conditions for parity-conserving spaces are provided by the following

Theorem

Consider a local relativistic quantum field theory. This theory is parity-conserving if the following conditions hold:

(a) Its fields are derived from geometry, i.e. are represented by quantised currents (in the sense of de Rham), and
(b) The theory is defined on a connected orientable differentiable manifold M^4.

Proof: Let $\tilde{M}$ be the covering space (3.13). We define on $\tilde{M}$ the following *parity-orientation operator*

$$\tilde{P} : (x,\epsilon) \to (x',\epsilon') \equiv (x^0,-\mathbf{x},\epsilon') \qquad \epsilon,\epsilon' \text{ fixed} \tag{5.1}$$

which is induced by the space-inversion symmetry P. (This re-definition of the parity operator is indispensable, since statements involving the orientation must obviously be related to $\tilde{M}$.) In particular, if M admits some orientation function of the type (3.5): $\epsilon(x,\Phi) = \epsilon_\Phi(x) \in \{-1,+1\}$, which is associated with the atlas $\{(U_i,\Phi_i)\}_{i\in I}$ on M, $\tilde{P}$ maps $\epsilon_{\Phi_i}(x)$ into $\epsilon_{\Phi_j}(x')$.

However, since in each point of M^4 the tangent space T_x is endowed with two orientations $\epsilon = 1$ and $\epsilon = -1$, $\tilde{P}$ splits into the following operators:

$$\begin{aligned} \tilde{P}^+ &: (x,+1) \to (x',+1) \\ &\ (x,-1) \to (x',-1) \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \tilde{P}^- &: (x,+1) \to (x',-1) \\ &\ (x,-1) \to (x',+1) \end{aligned} \tag{5.3}$$

where $x = (x^0,\mathbf{x})$ and $x' = (x^0,-\mathbf{x})$. That is, $\tilde{P}^+$ carries the orientation associated with $(x^0,\mathbf{x})$ into the image point $(x^0,-\mathbf{x})$ and analogously $\tilde{P}^-$ reverses the orientations when mapping $(x^0,\mathbf{x})$ into $(x^0,-\mathbf{x})$.

Thus we may replace equations (2.2)–(2.3) by the subsequent geometrical conditions (5.4)–(5.5). (We shall discuss only the case of a scalar field, since the space-inversion invariance condition does not affect the principle in the case of spinor fields.)

$$U(\tilde{P}^+)\,\tilde{\omega}(\tilde{\phi})\,U(\tilde{P}^+)^{-1} = \tilde{\omega}[(0,\tilde{P}^+)\tilde{\phi}] \tag{5.4}$$

$$U(\tilde{P}^-)\,\tilde{\omega}(\tilde{\phi})\,U(\tilde{P}^-)^{-1} = \tilde{\omega}[(0,\tilde{P}^-)\tilde{\phi}] \tag{5.5}$$

where $\tilde{\omega}$ and $\tilde{\phi}$ denote forms on $\tilde{M}^4$. Indeed, by virtue of (3.18) and (4.6), statements concerning forms on M^4 can be enunciated equivalently on $\tilde{M}^4$. Equations (5.4) and (5.5) must be supplemented by the following

Condition A

The quantised operator-valued currents $\tilde{\omega}[(0,\tilde{P}^+)\tilde{\phi}]$ and $\tilde{\omega}[(0,\tilde{P}^-)\tilde{\phi}]$ are identical.

Therefore, (5.4) and (5.5) are both given by the same expression

$$\tilde{\omega}[(0,\tilde{P}^{\pm})\tilde{\phi}] = +\int \tilde{\omega}(\tilde{x}_{\pm})\,\tilde{\phi}(-\tilde{x}_{\pm})\,d\tilde{x}_{\pm} \tag{5.6}$$

where $\tilde{x}_{\pm} \equiv (x,\pm\epsilon)$ and $-\tilde{x}_{\pm} \equiv (x,-\mathbf{x},\pm\epsilon)$.

The motivation of this condition is clear. Non-agreement of the fields (5.4) and (5.5) would lead to contradiction with the uniquely defined parity transformation law (2.3a) of scalar fields. This amounts to saying that the action of the unitary operators $U(\tilde{P}^+)$ and $U(\tilde{P}^-)$ on $\tilde{\omega}(\tilde{\phi})$ must be the same up to unitary equivalence. The expressions (5.4)–(5.5) together with condition A are equivalent to the properties (2.2) and (2.3). Indeed, the quantised differential form fields have been shown (according to Lemma 2) to generalise the averaged quantum fields of the type (2.3b) and (2.3c). These fields, defined on M, are either π-invariant (normal fields) or π-anti-invariant (twisted fields) on $\tilde{M}$. Thus the alleged equivalence is obtained if and only if $\tilde{P}$ is defined, according to (5.1), on elements $(x^0,\mathbf{x},\epsilon) \in \tilde{M}$.

According to Section 4, four different types of averaged quantum fields may be defined on M. Two types of even quantised forms are available if and only if M is orientable, namely, $\omega(\phi)$ and $\boldsymbol{\omega}(\boldsymbol{\phi})$ may be defined on some differentiable manifold which is orientable or not. By virtue of Section 4, fields of the type $\boldsymbol{\omega}(\phi)$ are not suitable. For similar reasons (refer to remark 7) 'normal' forms of the type $\boldsymbol{\omega}(\boldsymbol{\phi})$ on oriented manifolds are to be rejected. Note that fields of the kind $\omega(\phi)$ do not fit either, since ω and ϕ, having the same parity, do not define any current at all. This applies also to some of the aforementioned fields. Therefore, the only type of current to be considered is $\omega(\boldsymbol{\phi})$. These currents are defined on orientable as well as non-orientable manifolds.

The behaviour under the parity-orientation operators $\tilde{P}^{\pm}$ of the only admissible fields $\omega(\boldsymbol{\phi})$ is exhibited by

$$\omega(\boldsymbol{\phi}) \to \tilde{\omega}[(0,\tilde{P}^+)\tilde{\phi}] = \int \tilde{\omega}(\tilde{x}_+)\,\tilde{\phi}(-\tilde{x}_+)\,d\tilde{x}_+ \tag{5.7}$$

and

$$\omega(\phi) \to \tilde{\omega}[(0,\tilde{P}^-)\tilde{\phi}] = \int \tilde{\omega}(\tilde{x}_+)\tilde{\phi}(-\tilde{x}_-)\,d\tilde{x}_+$$
$$= -\int \tilde{\omega}(\tilde{x}_+)\tilde{\phi}(-\tilde{x}_+)\,d\tilde{x}_+ \tag{5.8}$$

(The change in sign in (5.7) and (5.8) is due to ϕ being a twisted 0-form.)

Lemma 3

Let M^4 be a connected C^k-manifold ($k \geqslant 1$) and let $\tilde{M}^4$ be its covering space. If $\tilde{P}^-$, defined by (5.3), exists on $\tilde{M}^4$, then M^4 cannot be orientable.

The proof of this Lemma is based on the following:

Lemma 4

If a manifold M^n is connected, two continuous systems of orientations have either the same or the opposite sign on the whole manifold.

Proof of Lemma 4: Let $\epsilon_1(x)$ and $\epsilon_2(x)$ be two continuous orientation functions and set:

$$\epsilon_3(x) = \frac{\epsilon_1(x)}{\epsilon_2(x)} = \epsilon_1(x)\epsilon_2(x) \qquad \forall x \in M$$

ϵ_3 being a third continuous orientation function, it takes its values in $\{+1,-1\}$, thus

$$\epsilon_3^{-1} : \{1,-1\} \to M$$

and, by virtue of the continuity of ϵ_3, ϵ_3^{-1} maps open (closed) sets in open (closed) sets. Now $\{+1\}$ is closed, but also open, since $C_{\{1,-1\}}\{1\} = \{-1\}$ is closed. Therefore $\epsilon_3^{-1}(+1) \subset M$ is closed and open at the same time. Since M is connected, the only open and closed sets of M are $\varnothing$ and M itself. Therefore:

(a) If

$$\epsilon_3^{-1}(+1) = M \Rightarrow \epsilon_3(x) = +1 \qquad \forall x \in M$$
$$\Rightarrow \epsilon_1(x) \text{ and } \epsilon_2(x)$$

both have the same sign: $\forall x \in M$; or

(b) If

$$\epsilon_3^{-1}(+1) = \varnothing \Rightarrow \nexists x \in M$$

such that $\epsilon_3(x) = +1$ but:

$$\epsilon_3 : M \to \{+1,-1\} \Rightarrow \epsilon_3(x) = -1 \qquad \forall x \in M$$
$$\Rightarrow \epsilon_1(x) \text{ and } \epsilon_2(x)$$

have opposite signs: $\forall x \in M$. The same reasoning holds for $\epsilon_3^{-1}(-1)$. This achieves the proof.

Now we can proceed to the

Proof of Lemma 3: Proof by contradiction. Let M be orientable, then Lemma 4 yields:

M is connected and orientable $\Leftrightarrow$

$$(\exists \epsilon_1, \epsilon_2)(\forall x \in M) : \epsilon_1(x) = +1 \quad \text{and} \quad \epsilon_2(x) = -1 \tag{5.9}$$

Now: (5.3) $\tilde{P}^- : (x, \pm 1) \rightarrow (x', \mp 1)$ means $\epsilon(x) = -\epsilon(x')$, $\forall$ orientation system ϵ; thus (5.10) holds also for the two special continuous orientation systems ϵ_1 and ϵ_2:

$$\epsilon_1(x) = +1 \qquad \epsilon_1(x') = -1$$
$$\epsilon_2(x) = -1 \qquad \epsilon_2(x') = +1$$

This obviously contradicts (5.9).

By virtue of Lemma 3, one has to distinguish between the following two cases:

(a) On a connected orientable manifold, the operator $\tilde{P}^-$ may not be defined, thus condition (5.5) becomes meaningless. Therefore the field $\omega(\boldsymbol{\phi})$ can only transform according to $\tilde{P}^+$, which, according to (5.4), ensures parity-conservation.

(b) On a connected non-orientable manifold, the parity is violated, since condition A cannot be satisfied by the field $\omega[\tilde{P}^- \boldsymbol{\phi}]$.

This achieves the proof of the theorem.

Corollary

Let M be an orientable disconnected manifold, such that the connected components M_i form a partition of M, i.e.

$$M = \bigcup_{i \in I} M_i \tag{5.10}$$

If condition

$$(\forall i)(\exists x_i = (x_i{}^0, \mathbf{x}_i) \in M_i) : (x_i{}^0, -\mathbf{x}_i) \in M_i \tag{5.11}$$

holds, then the parity is conserved for fields defined on M.

Proof: If condition (5.11) holds, then, by virtue of Lemma 3, the operator $\tilde{P}^-$ may not be defined on the covering spaces $\tilde{M}_i$ of the connected components M_i, since, for at least one $x_i \in M_i$, this operator is in contradiction with the orientability of M_i. This yields that $\tilde{P}^-$ cannot be defined on M, which in turn entails the condition (5.5) of parity invariance to be invalidated.

6. *Concluding Remarks*

Remark 8: The proof of the main theorem has been achieved on purely geometrical considerations. No use has been made of the explicit form of the Hamiltonian functional H, nor had the dynamical invariance property (2.4) $[H, U(P)] = 0$ to be taken into account in proving the theorem. Indeed, within a non-geometric framework of a quantum field theory, condition (2.4) is actually necessary to determine the matrix elements of the dynamical operator H (together with the operator equation of motion, $[H, \Phi(\phi)] = -i\Phi(\phi)$, the Hermiticity of H and the spectrum normalisation condition $H\Omega = 0$).

Thus it appears clear that, if the geometry is suitably chosen, the breakdown of the space-inversion symmetry P can only be due to the dynamics. Therefore, only the non-fulfilment of the dynamical invariance postulate (2.4) yields parity non-conservation.

Remark 9: Condition (b) of the theorem is always satisfied in the case of a conventional local relativistic quantum field theory, since such a theory is defined on $\mathbf{R}^4$. However, in order to obtain a P-invariant theory, the concept of field has to be modified according to condition (a) of the theorem. A closer inspection of the proof of our theorem displays furthermore that the orientability of the manifold constitutes a major argument. This point is entirely neglected within the framework of a conventional quantum field theory, where this property of the underlying universe is not exploited at all.

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